Noncommutative field theory

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Abstract

This article reviews the generalization of field theory to space-time with noncommuting coordinates, starting with the basics and covering most of the active directions of research. Such theories are now known to emerge from limits of M theory and string theory and to describe quantum Hall states. In the last few years they have been studied intensively, and many qualitatively new phenomena have been discovered, on both the classical and the quantum level.

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I. INTRODUCTION

Noncommutativity is an age-old theme in mathematics and physics. The noncommutativity of spatial rotations in three and more dimensions is deeply ingrained in us. Noncommutativity is the central mathematical concept expressing uncertainty in quantum mechanics, where it applies to any pair of conjugate variables, such as position and momentum. In the presence of a magnetic field, even momenta fail to mutually commute.

One can just as easily imagine that position measurements might fail to commute and describe this using noncommutativity of the coordinates. The simplest noncommutativity one can postulate is the commutation relation

$$[x^i, x^j] = i\theta^{ij}, \tag{1}$$

with a parameter θ which is an antisymmetric (constant) tensor of dimension (length)^2.

As has been realized independently many times, at least as early as 1947 (Snyder, 1947), there is a simple modification to quantum field theory obtained by taking the position coordinates to be noncommuting variables. Starting with a conventional field theory Lagrangian and interpreting the fields as depending on coordinates satisfying Eq. (1), one can follow the usual development of perturbative quantum field theory with surprisingly few changes, to define a large class of "noncommutative field theories."

It is on this class of theories that our review will focus. Until recently, such theories had not been studied very seriously. Perhaps the main reason for this is that postulating an uncertainty relation between position measurements will *a priori* lead to a nonlocal theory, with all of the attendant difficulties. A secondary reason is that noncommutativity of the space-time coordinates generally conflicts with Lorentz invariance, as is apparent in Eq. (1). Although it is not implausible that a theory defined using such coordinates could be effectively local on length scales longer than that of θ, it is harder to believe that the breaking of Lorentz invariance would be unobservable at these scales.

Nevertheless, one might postulate noncommutativity for a number of reasons. Perhaps the simplest is that it might improve the renormalizability properties of a theory at short distances or even render it finite. Without giving away too much of our story, we should say that this is of course not obvious *a priori* and a noncommutative theory might turn out to have the same or even worse short-distance behavior than a conventional theory.

Another motivation is the long-held belief that in quantum theories including gravity, space-time must change its nature at distances comparable to the Planck scale. Quantum gravity has an uncertainty principle which prevents one from measuring positions to better accuracies than the Planck length: the momentum and energy required to make such a measurement will itself modify the geometry at these scales (DeWitt, 1962). One might wonder if these effects could be modeled by a commutation relation such as Eq. (1).

A related motivation is that there are reasons to believe that any theory of quantum gravity will not be local in the conventional sense. Nonlocality brings with it deep conceptual and practical issues which have not been well understood, and one might want to understand them in the simplest examples first, before proceeding to a more realistic theory of quantum gravity.

This is one of the main motivations for the intense current activity in this area among string theorists. String theory is not local in any sense we now understand, and indeed has more than one parameter characterizing this nonlocality: in general, it is controlled by the larger of the Planck length and the "string length," the average size of a string. It was discovered by Connes *et al.* (1998) and Douglas and Hull (1998) that simple limits of M theory and string theory lead directly to noncommutative gauge theories, which appear far simpler than the original string theory yet keep some of this nonlocality.

One might also study noncommutative theories as interesting analogs of theories of more direct interest, such as Yang-Mills theory. An important point in this regard is that many theories of interest in particle physics are so highly constrained that they are difficult to study. For example, pure Yang-Mills theory with a definite simple gauge group has no dimensionless parameters with which to make a perturbative expansion or otherwise simplify the analysis. From this point of view it is quite interesting to find any sensible and nontrivial variants of these theories.

Now, physicists have constructed many, many variations of Yang-Mills theory in the search for regulated (UV finite) versions as well as more tractable analogs of the theory. A particularly interesting example in the present context is the twisted Eguchi-Kawai model (Eguchi and Nakayama, 1983; Gonzalez-Arroyo and Okawa, 1983), which in some of its forms, especially that of Gonzalez-Arroyo and Korthals Altes (1983), is a noncommutative gauge theory. This model was developed in the study of the large-N limit of Yang-Mills theory ('t Hooft, 1974) and we shall see that noncommutative gauge theories show many analogies to this limit (Filk, 1996; Minwalla *et al.*, 2000), suggesting that they should play an important role in the circle of ideas relating large-N gauge theory and string theory (Aharony *et al.*, 2000; Polyakov, 1987).

Noncommutative field theory is also known to appear naturally in condensed-matter theory. The classic example (though not always discussed using this language) is the theory of electrons in a magnetic field projected to the lowest Landau level, which is naturally thought of as a noncommutative field theory. Thus these ideas are relevant to the theory of the quantum Hall effect (Prange and Girvin, 1987), and indeed, noncommutative geometry has been found very useful in this context (Bellissard *et al.*, 1993). Most of this work has treated noninteracting electrons, and it seems likely that introducing field-theoretic ideas could lead to further progress.

It is interesting to note that despite the many physical motivations and partial discoveries we just recalled, noncommutative field theory and gauge theory were first clearly formulated by mathematicians (Connes and Rieffel, 1987). This is rather unusual for a theory of significant interest to physicists; usually, as with Yang-Mills theory, the flow goes in the other direction.

An explanation for this course of events might be found in the deep reluctance of physicists to regard a nonlocal theory as having any useful space-time interpretation. Thus, even when these theories arose naturally in physical considerations, they tended to be regarded only as approximations to more conventional local theories, and not as ends in themselves. Of course such sociological questions rarely have such pat answers and we shall not pursue this one further except to remark that, in our opinion, the mathematical study of these theories and their connection to noncommutative geometry has played an essential role in convincing physicists that these are not arbitrary variations on conventional field theory but indeed a new universality class of theory deserving study in its own right. Of course this mathematical work has also been an important aid to the more prosaic task of sorting out the possibilities, and it is the source for many useful techniques and constructions that we shall discuss in detail.

Having said this, it seems that the present trend is that the mathematical aspects appear less and less central to the physical considerations as time goes on. While it is too early to judge the outcome of this trend and it seems certain that the aspects which traditionally have benefited most from mathematical influence will continue to do so (especially, the topology of gauge-field configurations, and techniques for finding exact solutions), we have to some extent deemphasized the connections with noncommutative geometry in this review. This is partly to make the material accessible to a wider class of physicists, and partly because many excellent books and reviews cover the material from this point of view, starting with that of Connes (1994). Other reviews include those of Nekrasov (2000) focusing on classical solutions of noncommutative gauge theory, Konechny and Schwarz (2000) focusing on duality properties of gauge theory on a torus, and Varilly (1997) and Gracia-Bondia *et al.* (2001). We maintain one section which attempts to give an overview of aspects for which a more mathematical point of view is clearly essential.

Although many of the topics we shall discuss were motivated by and discovered in the context of string theory, we have also taken the rather unconventional approach of separating the discussion of noncommutative field theory from that of its relation to string theory, to the extent that this was possible. An argument against this approach is that the relation clarifies many aspects of the theory, as we hope will become abundantly clear upon reading Sec. V. However, it is also true that string theory is not a logical prerequisite for studying the theory, and we feel the approach we took better illustrates its internal self-consistency (and the points where this is still lacking). Furthermore, if we hope to use noncommutative field theory as a source of *new* insights into string theory, we need to be able to understand its physics without relying too heavily on the analogy. We also hope this approach will have the virtue of broader accessibility and perhaps help in finding interesting applications outside of string theory. Reviews with a more string-theoretic emphasis include that of Harvey (2001) which discusses solitonic solutions and their relations to string theory.

Finally, we must apologize to the many whose work we were not able to treat in the depth it deserved in this review, a sin we have tried to atone for by including an extensive bibliography.

II. KINEMATICS

A. Formal considerations

Let us start by defining noncommutative field theory in a somewhat pedestrian way, by proposing a configuration space and action functional from which we could either derive equations of motion or define a functional integral. We shall discuss this material from a more mathematical point of view in Sec. IV.

Conventions. Throughout the review we use the following notations: Latin indices $i, j, k, \ldots$ denote space-time indices, Latin indices from the beginning of the alphabet $a, b, \ldots$ denote commutative dimensions, Greek indices $\mu, \nu, \ldots$ enumerate particles, vertex operators, etc, while Greek indices from the beginning of the alphabet $\alpha, \beta, \ldots$ denote noncommutative directions.

In contexts where we simultaneously discuss a noncommuting variable or field and its commuting analog, we shall use the "hat" notation: x is the commuting analog to $\hat{x}$. However, in other contexts, we shall not use the hat.

1. The algebra

The primary ingredient in the definition is an associative but not necessarily commutative algebra, to be denoted $\mathcal{A}$. The product of elements a and b of $\mathcal{A}$ will be denoted ab, $a \cdot b$, or $a \star b$. This last notation (the "star product") has a special connotation, to be discussed shortly.

An element of this algebra will correspond to a configuration of a classical complex scalar field on a "space" M. Suppose first that $\mathcal{A}$ is commutative. The primary example of a commutative associative algebra is the algebra of complex-valued functions on a manifold M, with addition and multiplication defined pointwise: $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x)g(x)$. In this case, our definitions will reduce to the standard ones for field theory on M.

Although the mathematical literature is usually quite precise about the class of functions (continuous, smooth, etc.) to be considered, in this review we follow standard physical practice and simply consider all functions that arise in reasonable physical considerations, referring to this algebra as $\mathcal{A}(M)$ or (for reasons to be explained shortly) as "M_0". If more precision is wanted, for most purposes one can think of this as $C(M)$, the bounded continuous functions on the topological manifold M.

The most elementary example of a noncommutative algebra is Mat_n, the algebra of complex $n \times n$ matrices. Generalizations of this, which are almost as elementary, are the algebras $\mathrm{Mat}_n(C(M))$ of $n \times n$ matrices whose matrix elements are elements of $C(M)$, and with addition and multiplication defined according to the usual rules for matrices in terms of the addition and multiplication on $C(M)$. This algebra contains $C(M)$ as its center (take functions times the identity matrix in Mat_n).

Clearly elements of $\mathrm{Mat}_n(C(M))$ correspond to configurations of a matrix field theory. Just as one can gain some intuition about operators in quantum mechanics by thinking of them as matrices, this example already serves to illustrate many of the formal features of noncommutative field theory. In the remainder of this subsection we introduce the other ingredients we need to define noncommutative field theory in this familiar context.

To define a real-valued scalar field, it is best to start with $\mathrm{Mat}_n(C(M))$ and then impose a reality condition analogous to the reality of functions in $C(M)$. The most useful in practice is to take the Hermitian matrices $a = a^\dagger$, whose eigenvalues will be real (given suitable additional hypotheses). To do this for general $\mathcal{A}$, we would need an operation $a \to a^\dagger$ satisfying $(a^\dagger)^\dagger = a$ and (for $c \in \mathbb{C}$) $(ca)^\dagger = c^* a^\dagger$, in other words an antiholomorphic involution.

The algebra $\mathrm{Mat}_n(C(M))$ could also be defined as the tensor product $\mathrm{Mat}_n(\mathbb{C}) \otimes C(M)$. This construction generalizes to an arbitrary algebra $\mathcal{A}$ to define $\mathrm{Mat}_n(\mathbb{C}) \otimes \mathcal{A}$, which is just $\mathrm{Mat}_n(\mathcal{A})$ or $n \times n$ matrices with elements in $\mathcal{A}$. This algebra admits the automorphism group $GL(n,\mathbb{C})$, acting as $a \to g^{-1}ag$ (of course the center acts trivially). Its subgroup $U(n)$ preserves Hermitian conjugation and the reality condition $a = a^\dagger$. One sometimes refers to these as "$U(n)$ noncommutative theories," a bit confusingly. We shall refer to them as *rank n theories.*

In the rest of the review, we shall mostly consider noncommutative associative algebras which are related to the algebras $\mathcal{A}(M)$ by deformation with respect to a parameter θ, as we shall define shortly. Such a deformed algebra will be denoted by M_θ, so that $M_0 = \mathcal{A}(M)$.

2. The derivative and integral

A noncommutative field theory will be defined by an action functional of fields $\Phi, \phi, \varphi, \ldots$ defined in terms of the associative algebra $\mathcal{A}$ (it could be elements of $\mathcal{A}$, or vectors in some representation thereof). Besides the algebra structure, to write an action we shall need an integral $\int \mathrm{Tr}$ and derivatives ∂_i. These are linear operations satisfying certain formal properties:

(a) The derivative is a derivation on $\mathcal{A}$, $\partial_i(AB) = (\partial_i A)B + A(\partial_i B)$. With linearity, this implies that the derivative of a constant is zero.

(b) The integral of the trace of a total derivative is zero, $\int \mathrm{Tr}\ \partial_i A = 0$.

(c) The integral of the trace of a commutator is zero, $\int \mathrm{Tr}\ [A, B] \equiv \int \mathrm{Tr}\ (A \cdot B - B \cdot A) = 0$.

A candidate derivative ∂_i can be written using an element $d_i \in \mathcal{A}$; let $\partial_i A = [d_i, A]$.

Derivations that can be written in this way are referred to as inner derivations, while those that cannot are outer derivations.

We denote the integral as $\int \mathrm{Tr}$, as it turns out that, for general noncommutative algebras, one cannot separate the notations of trace and integral. Indeed, one normally uses either the single symbol Tr (as is done in mathematics) or $\int$ to denote this combination; we do not follow this convention here only to aid the uninitiated.

We note that just as condition (b) can be violated in conventional field theory for functions that do not fall off at infinity, leading to boundary terms, condition (c) can be violated for general operators, leading to physical consequences in noncommutative theory which we shall discuss.

B. Noncommutative flat space-time

After $\mathrm{Mat}_n(C(M))$, the next simplest example of a noncommutative space is the one associated with the algebra $\mathbb{R}^d_\theta$ of all complex linear combinations of products of d variables $\hat{x}^i$ satisfying

$$[x^i, x^j] = i\theta^{ij}. \tag{2}$$

The i is present because the commutator of Hermitian operators is anti-Hermitian. As in quantum mechanics, this expression is the natural operator analog of the Poisson bracket determined by the tensor θ^{ij}, the *Poisson tensor* or noncommutativity parameter.

By applying a linear transformation to the coordinates, one can bring the Poisson tensor to canonical form. This form depends only on its rank, which we denote as $2r$. We keep this general as one often discusses partially noncommutative spaces, with $2r < d$.

A simple set of derivatives ∂_i can be defined by the relations

$$\partial_i x^j \equiv \delta_i^j \tag{3}$$

$$[\partial_i, \partial_j] = 0 \tag{4}$$

and the Leibniz rule. This choice also determines the integral uniquely (up to overall normalization), by requiring that $\int \partial_i f = 0$ for any f such that $\partial_i f \neq 0$.

We shall occasionally generalize Eq. (4) to

$$[\partial_i, \partial_j] = -i\Phi_{ij}, \tag{5}$$

to incorporate an additional background magnetic field.

Finally, we shall require a metric, which we shall take to be a constant symmetric tensor g_{ij}, satisfying $\partial_i g_{jk} = 0$. In many examples we take this to be $g_{ij} = \delta_{ij}$, but note that one cannot bring both g_{ij} and θ^{ij} to canonical form simultaneously, as the symmetry groups preserved by the two structures, $O(n)$ and $Sp(2r)$, are different. At best one can bring the metric and the Poisson tensor to the following form:

$$g = \textstyle\sum_{\alpha=1}^{r} dz_\alpha d\bar{z}_\alpha + \sum_b dy_b^2; \qquad \theta = \frac{1}{2}\sum_\alpha \theta_\alpha\, \partial_{\bar{z}_\alpha} \wedge \partial_{z_\alpha}; \qquad \theta_a > 0.$$

Here $z_\alpha = q_\alpha + ip_\alpha$ are convenient complex coordinates. In terms of p, q, y the metric and the commutation relations Eq. (6) read as

$$[y_a, y_b] = [y_b, q_\alpha] = [y_b, p_\alpha] = 0, \quad [q_\alpha, p_\beta] = i\theta_\alpha\ \delta_{\alpha\beta} \qquad ds^2 = dq_\alpha^2 + dp_\alpha^2 + dy_b^2.$$

1. Symmetries of $\mathbb{R}^d_\theta$

An infinitesimal translation $x^i \to x^i + a^i$ on $\mathbb{R}^d_\theta$ acts on functions as $\delta\phi = a^i\partial_i\phi$. For the noncommuting coordinates x^i, these are formally inner derivations, as

$$\partial_i f = [-i(\theta^{-1})_{ij} x^j, f]. \tag{6}$$

One obtains global translations by exponentiating these,

$$f(x^i + \varepsilon^i) = e^{-i\theta_{ij}\varepsilon^i x^j} f(x) e^{i\theta_{ij}\varepsilon^i x^j}. \tag{7}$$

In commutative field theory, one draws a sharp distinction between translation symmetries (involving the derivatives) and internal symmetries, such as $\delta\phi = [A, \phi]$. We see that in noncommutative field theory, there is no such clear distinction, and this is why one cannot separately define integral and trace.

One often uses only $[\partial_i, f]$ and if so, Eq. (6) can be simplified further to the operator substitution $\partial_i \to -i(\theta^{-1})_{ij}x^j$. This leads to derivatives satisfying Eq. (5) with $\Phi_{ij} = -(\theta^{-1})_{ij}$.

The $Sp(2r)$ subgroup of the rotational symmetry $x^i \to R^i_j\ x^j$ which preserves θ, $R^{i'}_i R^{j'}_j \theta_{i'j'} = \theta_{ij}$ can be obtained similarly, as

$$f(R^i_j\ x^j) = e^{-iA_{ij}x^i x^j}\ f(x)\ e^{iA_{ij}x^i x^j} \tag{8}$$

where $R = e^{iL}$, $L^i_j = A_{kj}\theta^{ik}$, and $A_{ij} = A_{ji}$. Of course only the $U(r)$ subgroup of this will preserve the Euclidean metric.

After considering these symmetries, we might be tempted to go on and conjecture that

$$\delta\phi = i[\phi, \epsilon] \tag{9}$$

for any ϵ is a symmetry of $\mathbb{R}^d_\theta$. However, although these transformations preserve the algebra structure and the trace,[1] they do not preserve the derivatives. Nevertheless they are important and will be discussed in detail below.

2. Plane-wave basis and dipole picture

One can introduce several useful bases for the algebra $\mathbb{R}^d_\theta$. For discussions of perturbation theory and scattering, the most useful basis is the plane-wave basis, which consists of eigenfunctions of the derivatives:

$$\partial_i e^{ikx} = ik_i e^{ikx}. \tag{10}$$

The solution e^{ikx} of this linear differential equation is the exponential of the operator $ik \cdot x$ in the usual operator sense.

The integral can be defined in this basis as

$$\int \mathrm{Tr}\ \ e^{ikx} = \delta_{k,0} \tag{11}$$

where we interpret the delta function in the usual physical way (for example, its value at zero represents the volume of physical space).

More interesting is the interpretation of the multiplication law in this basis. This is easy to compute in the plane-wave basis, by operator reordering:

$$e^{ikx} \cdot e^{ik'x} = e^{-\frac{i}{2}\theta^{ij}k_i k'_j} e^{i(k+k')\cdot x}. \tag{12}$$

The combination $\theta^{ij}k_i k'_j$ appearing in the exponent comes up very frequently, and a standard and convenient notation for it is

$$k \times k' \equiv \theta^{ij}k_i k'_j = k \times_\theta k',$$

[1] Assuming certain conditions on ϵ and ϕ

the latter notation being used to stress the choice of Poisson structure.

We can also consider

$$e^{ikx} \cdot f(x) \cdot e^{-ikx} = e^{-\theta^{ij} k_i \partial_j} f(x) = f(x^i - \theta^{ij} k_j). \tag{13}$$

Multiplication by a plane wave translates a general function by $x^i \to x^i - \theta^{ij} k_j$. This exhibits the nonlocality of the theory in a particularly simple way and gives rise to the principle that large momenta will lead to large nonlocality.

A simple picture can be made of this nonlocality (Bigatti and Susskind, 2000; Sheikh-Jabbari, 1999) by imagining that a plane wave corresponds not to a particle (as in commutative quantum field theory) but instead to a "dipole," a rigid oriented rod whose extent is proportional to its momentum:

$$\Delta x^i = \theta^{ij} p_j. \tag{14}$$

If we postulate that dipoles interact by joining at their ends, and grant the usual quantum field theory relation $p = \hbar k$ between wave number and momentum, the rule Eq. (13) follows immediately. See Fig. 1.

3. Deformation, operators, and symbols

There is a sense in which $\mathbb{R}^d_\theta$ and the commutative algebra of functions $C(\mathbb{R}^d)$ have the same topology and the same "size," notions we shall keep at an intuitive level. In the physical applications, it will turn out that θ is typically a controllable parameter, which one can imagine increasing from zero to go from commutative to noncommutative (this does not imply that the physics is continuous in this parameter, however). These are all reasons to study the relation between these two algebras more systematically.

There are a number of ways to think about this relation. If θ is a physical parameter, it is natural to think of $\mathbb{R}^d_\theta$ as a deformation of $\mathbb{R}^d$. A deformation M_θ of $C(M)$ is an algebra with the same elements and addition law (it is the same considered as a vector space) but a different multiplication law, which reduces to that of $C(M)$ as a (multi-)parameter θ goes to zero. This notion was introduced by Bayen *et al.* (1978) as an approach to quantization, and has been much studied since, as we shall discuss in Sec. IV. Such a deformed multiplication law is often denoted $f \star g$ or *star product* to distinguish it from the original pointwise multiplication of functions.

This notation has a second virtue, which is that it allows us to work with M_θ in a way that is somewhat more forgiving of ordering questions. Namely, we can choose a linear map S from M_θ to $C(M)$, $\hat{f} \mapsto S[\hat{f}]$, called the *symbol* of the operator. We then represent the original operator multiplication in terms of the star product of symbols as

$$\hat{f}\hat{g} = S^{-1}[\ S[\hat{f}] \star S[\hat{g}]\ ]. \tag{15}$$

One should recall that the symbol is not "natural" in the mathematical sense: there could be many valid definitions of S, corresponding to different choices of operator ordering prescription for S^{-1}.

A convenient and standard choice is the Weyl ordered symbol. The map S, defined as a map taking elements of $\mathbb{R}^d_\theta$ to $\mathcal{A}(\mathbb{R}^d)$ (functions on momentum space), and its inverse, are

$$f(k) \equiv S[\hat{f}](k) = \frac{1}{(2\pi)^{n/2}} \int \mathrm{Tr}\ e^{-ik\hat{x}} \hat{f}(\hat{x}) \tag{16}$$

$$\hat{f}(\hat{x}) = S^{-1}[f] = \frac{1}{(2\pi)^{n/2}} \int d^n k\ e^{ik\hat{x}} f(k). \tag{17}$$

Formally these are inverse Fourier transforms, but the first expression involves the integral Eq. (11) on $\mathcal{A}(\mathbb{R}_\theta)$, while the second is an ordinary momentum-space integral.

One can get the symbol in position space by performing a second Fourier transform; e.g.,

$$S[\hat{f}](x) = \frac{1}{(2\pi)^n} \int d^n k \int \mathrm{Tr}\ e^{ik(x-\hat{x})} \hat{f}(\hat{x}). \tag{18}$$

We shall freely assume the usual Fourier relation between position and momentum space for the symbols, while being careful to say (or denote by standard letters such as x and k) which we are using.

The star product for these symbols is

$$e^{ikx} \star e^{ik'x} = e^{-\frac{i}{2}\theta^{ij} k_i k'_j} e^{i(k+k')\cdot x}. \tag{19}$$

Of course all of the discussion in B.2 above still applies, as this is only a different notation for the same product Eq. (12).

Another special case that often comes up is

$$\int \mathrm{Tr}\ f \star g = \int \mathrm{Tr}\ fg. \tag{20}$$

4. The noncommutative torus

Much of this discussion applies with only minor changes to define $\mathbf{T}^d_\theta$, the algebra of functions on a noncommutative torus.

To obtain functions on a torus from functions on $\mathbb{R}^d$, we would need to impose a periodicity condition, say $f(x^i) = f(x^i + 2\pi n^i)$. A nice algebraic way to phrase this is to instead define $\mathbf{T}^d_\theta$ as the algebra of all sums of products of arbitrary integer powers of a set of d variables U_i, satisfying

$$U_i U_j = e^{-i\theta^{ij}} U_j U_i. \tag{21}$$

The variable U_i takes the place of e^{ix^i} in our previous notation, and the derivation of the Weyl algebra from Eq. (1) is familiar from quantum mechanics. Similarly, we take

$$[\partial_i, U_j] = i\delta_{ij} U_j,$$

and

$$\int \mathrm{Tr}\ U_1^{n_1} \ldots U_d^{n_d} = \delta_{\vec{n},0}.$$

There is much more to say in this case about the topological aspects, but we postpone this to Sec. IV.

III. NONCOMMUTATIVE QUANTUM FIELD THEORY

IV. MATHEMATICAL ASPECTS

V. RELATIONS TO STRING AND M THEORY

VI. EXAMPLE OF TABLE

Their results are summarized in Table I.

VII. CONCLUSIONS

Field theory can be generalized to space-time with noncommuting coordinates. Much of the formalism is very parallel to that of conventional field theory and especially with the large-N limit of conventional field theory. Although not proven, it appears that quantum noncommutative field theories under certain restrictions (say with spacelike noncommutativity and some supersymmetry) are renormalizable and sensible.

Their physics is similar enough to conventional field theory to make comparisons possible, and different enough to make them interesting. To repeat some of the highlights, we found that noncommutative gauge symmetry includes space-time symmetries, that nonsingular soliton solutions exist in higher-dimensional scalar field theory and in noncommutative Maxwell theory, that UV divergences can be transmuted into new IR effects, and that noncommutative gauge theories can have more dualities than their conventional counterparts.

Much of our knowledge of conventional field theory still awaits a noncommutative counterpart. Throughout the review many questions were left open, such as the meaning of the IR divergences found in Sec. IV.C, the potential nonperturbative role of the solitons and instantons of Sec. III, the meaning of the high-energy behavior discussed in Sec. III.D, and the high-temperature behavior.

One central problem is to properly understand the renormalization group. Even if one can directly adapt existing RG technology, it seems very likely that theories with such a different underlying concept of space and time will admit other and perhaps more suitable formulations of the RG. This might lead to insights into nonlocality of the sort hoped for in the introduction. Questions about the existence of quantized noncommutative theories could then be settled by using the RG starting with a good regulated nonperturbative definition of the theory, perhaps that of Ambjorn *et al.* (1999) or perhaps along other lines as discussed in Sec. VI.A.

The techniques of exactly solvable field theory, which are so fruitful in two dimensions, await possible noncommutative generalization. These might be particularly relevant for the quantum Hall application.

It is not impossible that noncommutative field theory has some direct relevance for particle physics phenomenology, or possible relevance in the early universe. Possible signatures of noncommutativity in QED and the standard model are discussed by several authors[2] who work with a general extension of the standard model allowing for Lorentz violation [see Kostelecky (2001) and references therein] and argue that atomic clock-comparison experi-

[2] See, for example, Arfaei and Yavartanoo (2000); Baek *et al.* (2001); Carroll *et al.* (2001); Hewett *et al.* (2000); Mathews (2001); Mazumdar and Sheikh-Jabbari (2000); Mocioiu *et al.* (2000).

ments lead to a bound in the QED sector of $|\theta| < (10\text{TeV})^{-2}$. A noncommutative "brane world" scenario is developed by Pilo and Riotto (2001), and cosmological applications are discussed by (Alexander and Magueijo, 2001; Chu *et al.*, 2000).

This motivation as well as the motivation mentioned in the introduction of modeling position-space uncertainty in quantum gravity might be better served by Lorentz-invariant theories, and in pursuing the second of these motivations it has been suggested by Doplicher (2001); Doplicher *et al.* (1994) that such theories could be defined by treating the noncommutativity parameter θ as a dynamical variable. The space-time stringy uncertainty principle of Yoneya (1987) leads to related considerations (Yoneya, 2001).

While we hope that our discussion has demonstrated that noncommutative field theory is a subject of intrinsic interest, at present its primary physical application stems from the fact that it emerges from limits of M theory and string theory, and it seems clear at this point that the subject will have lasting importance in this context. So far its most fruitful applications have been to duality and to the understanding of solitons and branes in string theory. It is quite striking how much structure which had been considered "essentially stringy" is captured by these much simpler theories.

Noncommutativity enters into open string theory essentially because open strings interact by joining at their ends, and the choice of one or the other of the two ends corresponds formally to acting on the corresponding field by multiplication on the left or on the right; these are different. This is such a fundamental level that it has long been thought that noncommutativity should be central to the subject. So far, the developments we discussed look like a very promising start towards realizing this idea. Progress is also being made on the direct approach, through string field theory based on noncommutative geometry, and we believe that many of the ideas we discussed will reappear in this context.

Whether noncommutativity is a central concept in the full string or M theory is less clear. Perhaps the best reason to think this is that it appears so naturally from the definition of M(atrix) theory, which can include all of M theory in certain backgrounds. On the other hand, this also points to the weakness of our present understanding: these are very special backgrounds. We do not now have formulations of M theory in general backgrounds; this includes the backgrounds of primary physical interest with four observable dimensions. A related point is that in string theory, one thinks of the background as defined within the gravitational or closed string sector, and the role of noncommutativity in this sector is less clear.

An important question in noncommutative field theory is to what extent the definitions can be generalized to spaces besides Minkowski space and the torus, which are not flat. D-brane constructions in other backgrounds analogous to what we have discussed for flat space seem to lead to theories with finitely many degrees of freedom, as in Alekseev *et al.* (1999). It might be that noncommutative field theories can arise as large-N limits of these models, but at present this is not clear.

Even for group manifolds and homogeneous spaces, where mathematical definitions exist, the physics of these theories is not clear and deserves more study. As we discussed in Sec. VI, there are many more interesting noncommutative algebras arising from geometric constructions, which would be interesting test cases as well.

At the present state of knowledge, it is conceivable that, contrary to our intuition from the study of both gravity and perturbative string theory, special backgrounds such as flat space, anti-de Sitter space, orbifolds, and perhaps others, which correspond in M theory to simple gauge theories and noncommutative gauge theories, play a preferred role in the theory, and that all others will be derived from these. In this picture, the gravitational or closed string degrees of freedom would be derived from the gauge theory or open string theory, as has been argued to happen at substringy distances, in M(atrix) theory and in the AdS/CFT correspondence.

If physically realistic backgrounds could be derived this way, then this might be a satisfactory outcome. It would radically change our viewpoint on space-time and might predict that many backgrounds that would be acceptable solutions of gravity are in fact not allowed in M theory. It is far too early to judge this point, however, and it seems to us that at present such hopes are founded more on our lack of understanding of M theory in general backgrounds than on anything else. Perhaps noncommutative field theories in more general backgrounds, or in a more background-independent formulation, will serve as useful analogs to M/string theory for this question as well.

In any case, our general conclusion has to be that the study of noncommutative field theory, as well as the more mysterious theories which have emerged from the study of superstring duality (a few of which we mentioned in Sec. VII.G has shown that field theory is a much broader concept than had been dreamed of even a few years ago. It surely has many more surprises in store for us, and we hope this review will stimulate the reader to pick up and continue the story.

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Figures

FIG. 1 The interaction of two dipoles.

FIG. 2 Double line notation and phase factors.

FIG. 3 The contraction $(\mathrm{Tr}\ \phi^m)(\mathrm{Tr}\ \phi^n) \leftrightarrow \mathrm{Tr}\ \phi^{m+n-2}$.

FIG. 4 Tadpoles come with no phase factor.

FIG. 5 A non-planar diagram.

FIG. 6 T-duality to an anisotropic torus.

Tables

TABLE I Charge transfer times for Ar monolayers with and without selected spacer layers on Ru(0001), taken from Keller and Stichler (1998). The * indicates which layer is being probed.

Sample	Charge transfer time (fs)
Ar^*/Ru(0001)	1-2
Ar^*/O/Ru(0001)	3-4
Ar^*/CO/Ru(0001)	8
Ar^*/Xe/Ru(0001)	12
Ar/Ar^*/Xe/Ru(0001)	8
Ar^*/Ar/Xe/Ru(0001)	>50